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From Green's Theorem to the Shoelace Formula: An Application to the Calculation of the Area of the State of Tocantins

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Abstract

This article develops a deduction of the Shoelace Formula from Green's Theorem, highlighting the relationships between determinants, line integrals, and the calculation of areas of polygonal regions. First, the geometric interpretation of determinants is discussed as a tool for area calculation, establishing connections between concepts from Analytic Geometry and Vector Calculus. The Shoelace Formula is then deduced by parametrizing the edges of a simple polygon and applying Green's Theorem. As an application, the method is used to estimate the area of the State of Tocantins by discretizing its boundary into points in the Cartesian plane, yielding a value close to the officially reported area.

Keywords: Shoelace Formula; Green's Theorem; Determinants; Area Calculation; Mathematics Teaching.

Contents

1 Introduction

The calculation of areas is one of the most traditional topics in Mathematics and is present at different stages of school education. From the early years of Basic Education, students come into contact with procedures for determining the areas of plane figures, initially through formulas associated with elementary shapes and, later, through more general approaches

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involving coordinates, geometric transformations, and algebraic concepts. In this context, the study of areas fosters connections between different fields of Mathematics, making it possible to establish links between Geometry, Algebra, and Analysis.

Among the concepts closely related to area calculation, determinants stand out. Moreover to their importance in solving linear systems and in various applications of Linear Algebra, determinants admit a geometric interpretation associated with the measurement of areas and volumes. In particular, the absolute value of the determinant of a second-order matrix can be interpreted as the area of a parallelogram generated by two vectors in the plane, a result widely discussed in Linear Algebra textbooks [1, 5, 7].

Although the geometric interpretation of determinants is widely discussed in the Linear Algebra literature, its applications to the calculation of polygonal areas are not always explored in depth in materials intended for Basic Education. In particular, methods that allow the area of a polygonal region to be determined directly from the coordinates of its vertices are often presented in an operational manner, without a detailed discussion of their mathematical foundations.

This observation became especially evident from the analysis of an instructional sequence made available by the Tocantins State Department of Education (SEDUC-TO), in which the calculation of polygonal areas was carried out using a procedure based exclusively on the coordinates of the vertices. This procedure, also described by Silva [11], provides a simple and efficient way to determine the areas of convex polygons. However, the mathematical justification of the method and its connections with broader mathematical concepts are not discussed.

In this context, an interest arose in investigating the theoretical foundations of this procedure, known in the literature as the Shoelace Formula. The study revealed a rich connection between determinants, area calculation, and line integrals, showing how the formula can be obtained through an application of Green's Theorem. These relationships illustrate how concepts traditionally addressed at different educational levels can be articulated within a unifying mathematical perspective.

This article originates from an educational product developed within the dissertation *Matrices, Linear Systems, and Determinants in Basic Education: Concepts and Applications*, produced in the context of the National Professional Master's Program in Mathematics (PROFMAT). Its objective is to present the Shoelace Formula through its relationship with Green's Theorem, highlighting the geometric interpretations involved and illustrating its application through the estimation of the area of the State of Tocantins. In doing so, the work seeks to provide mathematics teachers with material that promotes a deeper understanding of the mathematical foundations underlying a method frequently used in educational contexts.

Beyond its mathematical interest, the approach offers opportunities to discuss connections among topics that are usually taught separately in school and undergraduate curricula. By relating determinants, analytic geometry, line integrals, and area calculation, the proposed discussion contributes to a more integrated view of Mathematics and highlights the potential of interdisciplinary approaches in Mathematics Education.

The application developed in this work takes as its object of study the geographic boundary of the State of Tocantins, making it possible to associate abstract mathematical concepts with a context close to the reality of students and teachers in the region. In this sense, the investigation contributes to bringing together concepts from Linear Algebra, Analytic Geometry, and Vector Calculus through a concrete situation, favoring a contextualized approach aligned with the appreciation of regional themes.

Although the Shoelace Formula is often presented as an algorithmic procedure for calculating the area of simple polygons, its mathematical foundation is not always explored in teaching materials. Works such as those by Braden [4], Bourke [3], and Santos [10] highlight the efficiency and elegance of the method, while Vector Calculus texts, such as Thomas, Weir, and Hass [13], make it possible to understand the formula as a natural consequence of Green's Theorem. From this perspective, the present work seeks to bring these different approaches together, emphasizing the relationships among determinants, analytic geometry, line integrals, and area calculation.

The distinguishing feature of this study lies in presenting an accessible exposition of the Shoelace Formula based on concepts from Vector Calculus, preserving mathematical rigor while emphasizing its didactic potential. Furthermore, the application of the method to estimate the area of the State of Tocantins illustrates how abstract mathematical concepts can be used to investigate problems related to the regional context, fostering an integrated approach between theory and application.

The article is organized as follows. Section 2 presents the geometric interpretation of determinants through the calculation of the area of triangles in the Cartesian plane, establishing results that will serve as a basis for subsequent discussions. Section 3 develops the Shoelace Formula through geometric arguments and subsequently presents its connection with Green's Theorem, culminating in an application to estimate the area of the State of Tocantins. Finally, Section 4 presents the concluding remarks and some reflections on the pedagogical potential of the approach developed.

2 Area of a Triangle and a Geometric Interpretation of Determinants

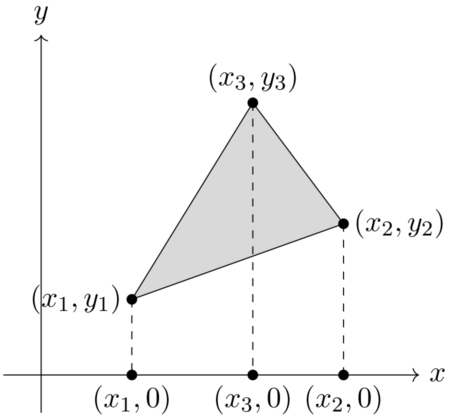
As discussed in the introduction, determinants play an important role not only in Linear Algebra but also in Geometry. One of their best-known geometric interpretations is related to area calculation in the Cartesian plane. In particular, the determinant of a second-order matrix can be interpreted as the signed area of the parallelogram generated by two vectors in the plane, a result widely explored in Linear Algebra textbooks [1, 5, 7].

From this interpretation, it is possible to obtain formulas for calculating the areas of polygonal figures using only the coordinates of their vertices. As a starting point for the discussions developed in this work, we present a formula for the area of a triangle whose proof highlights the close relationship between determinants and geometric concepts.

Proposition 2.1. *Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of a triangle in the Cartesian plane. Then its area is given by*

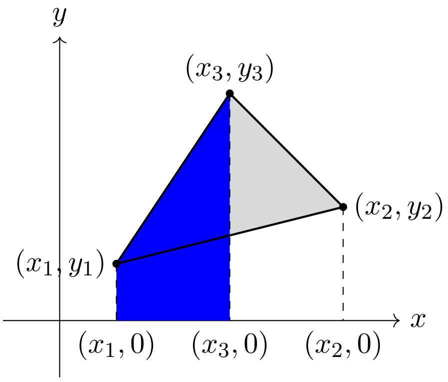
$$Area = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Proof. Without loss of generality, suppose that $x_1 \leq x_3 \leq x_2$ and that (x_3, y_3) lies above the line segment connecting (x_1, y_1) and (x_2, y_2).

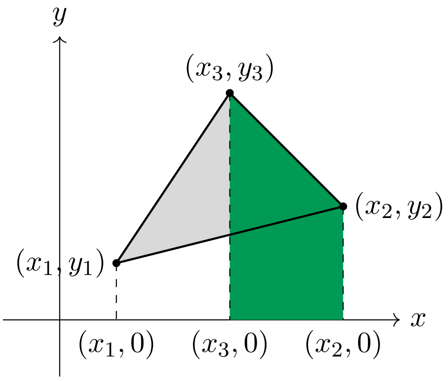


By projecting the vertices of the triangle onto the x-axis, we obtain three trapezoids whose areas allow us to decompose the area of the triangle in terms of elementary figures.

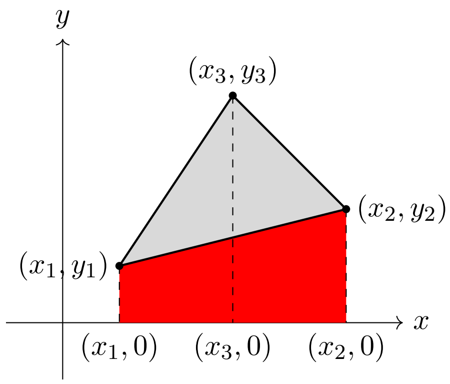
. Trapezoid 1: $(x_1, 0), (x_1, y_1), (x_3, y_3), (x_3, 0)$



. Trapezoid 2: $(x_3, 0), (x_3, y_3), (x_2, y_2), (x_2, 0)$



. Trapezoid 3: $(x_1, 0), (x_1, y_1), (x_2, y_2), (x_2, 0)$



Using the classical formula for the area of a trapezoid, that is, $A = \frac{(B+b)h}{2}$, where B and b represent the parallel bases and h the height, we obtain

- Area of Trapezoid 1: $A_{\tau_1} = \dfrac{1}{2}(y_3 + y_1)(x_3 - x_1)$
- Area of Trapezoid 2: $A_{\tau_2} = \dfrac{1}{2}(y_3 + y_2)(x_2 - x_3)$
- Area of Trapezoid 3: $A_{\tau_3} = \dfrac{1}{2}(y_2 + y_1)(x_2 - x_1)$

Note that the union of trapezoids τ_1 and τ_2 contains exactly the desired triangular region plus trapezoid τ_3. Thus, $A_{\tau_1} + A_{\tau_2} = A_\Delta + A_{\tau_3}$, from which it follows that $A_\Delta = A_{\tau_1} + A_{\tau_2} - A_{\tau_3}$. In other words, we have

$$\begin{aligned}
A_\Delta &= \frac{1}{2}\left[(y_3 + y_1)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) - (y_2 + y_1)(x_2 - x_1)\right] \\
&= \frac{1}{2}[\cancel{y_3x_3} - y_3x_1 + y_1x_3 - \cancel{y_1x_1} + y_3x_2 - \cancel{y_3x_3} + \cancel{y_2x_2} - y_2x_3 - \cancel{y_2x_2} + y_2x_1 - y_1x_2 + \cancel{y_1x_1}] \\
&= \frac{1}{2}[x_1y_2 + x_2y_3 + x_3y_1 - (x_1y_3 + x_2y_1 + x_3y_2)] \\
&= \frac{1}{2}\left(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2\right) \\
&= \frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.
\end{aligned}$$

Since the area is always non-negative, we conclude that

$$A_\Delta = \frac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|.$$

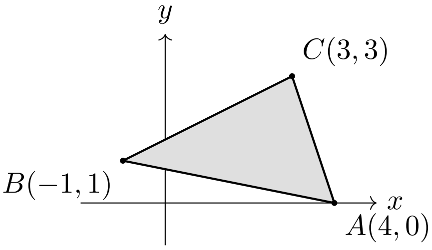
□

The previous result shows that the area of a triangle can be calculated directly from the coordinates of its vertices. More than a convenient formula, this result highlights the geometric interpretation of determinants as signed area measures, a concept that will be fundamental for the derivation of the Shoelace Formula presented in the following sections.

Note that the expression obtained in **Proposition 2.1** can take positive or negative values, depending on the order in which the triangle's vertices are considered. When the vertices are traversed counterclockwise, the determinant is positive; when traversed clockwise, the determinant is negative. Thus, the absolute value of the determinant gives the geometric area of the triangle, while its sign indicates the orientation of the vertices. To illustrate this geometric interpretation of determinants, we present below an example of calculating the area of a triangle in the Cartesian plane from the coordinates of its vertices.

To consolidate the geometric interpretation of the determinant as an area measure, we present below an example involving the calculation of the area of a triangle in the Cartesian plane using determinants. This procedure highlights the relationship between the coordinates of the vertices and the resulting value obtained for the area of the figure.

Example 2.2. We will determine the area of the triangular region whose vertices are the points $A(4,0)$, $B(-1,1)$, and $C(3,3)$. First, we represent the triangle in the Cartesian plane.



Applying the formula established in the previous proposition, we obtain

$$\begin{aligned} A_\Delta &= \frac{1}{2}\begin{vmatrix} 4 & 0 & 1 \\ -1 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} \\ &= \frac{1}{2}\left|4(1\cdot 1-3\cdot 1)+3((-1)\cdot 1-1\cdot 3)+1((-1)\cdot 3-1\cdot 3)\right| \\ &= \frac{1}{2}|-14| \\ &= 7. \end{aligned}$$

Therefore, the area of the triangular region is $A_\Delta = 7$ a.u.

Note that the determinant involved in the previous example can take positive or negative values, depending on the order in which the vertices are considered. Although the absolute value is necessary to obtain the geometric area of the triangular region, the sign of the determinant contains relevant information about the orientation of the path taken along the vertices. This observation naturally leads to the concept of signed area, a fundamental tool for the study of polygon areas and for the derivation of the Shoelace Formula.

2.1 Signed Area

The notion of signed area plays a central role in different areas of Mathematics, especially in Analytic Geometry, Linear Algebra, and Vector Calculus. As discussed by Anton [1], Larson [7], and Hefez [5], the sign of the determinant is directly associated with the orientation of the vectors that define it, making it possible to distinguish counterclockwise paths from clockwise ones.

In the Cartesian plane, this interpretation establishes a natural convention: paths traversed counterclockwise are associated with positive values of signed area, while paths traversed clockwise receive negative values. Thus, signed area simultaneously carries geometric information and information about the orientation of the boundary, an aspect that will be essential in generalizing to the calculation of polygon areas [4, 13].

Definition 2.3. Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be three distinct points in the Cartesian plane. The **signed area** of triangle ABC is defined as the real number

$$A_o(ABC) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The geometric area of the triangle is obtained by

$$A(ABC) = |A_o(ABC)|.$$

The orientation of the triangle is said to be *positive* when the vertices are traversed counterclockwise and *negative* when traversed clockwise.

Remark 2.4. The signed area does not depend on the geometric shape of the triangle, but on the order of the vertices chosen for its algebraic representation. Thus, cyclic permutations of the vertices preserve the value of A_o, while reversing the order of the vertices only changes the sign of the expression.

In **Example 2.2**, for the triangle with vertices $A(4, 0)$, $B(-1, 1)$, and $C(3, 3)$, direct application of the determinant expression shows that the value obtained for the area depends on the order in which the vertices are taken. When traversed in the order $A \to B \to C$, a positive quantity is obtained, while reversing this order produces the same value in absolute terms, but with a negative sign. This observation confirms, in a concrete way, that the determinant is not limited to calculating the geometric area, but also incorporates algebraic information associated with the orientation of the vertices in the Cartesian plane.

The definition of signed area allows composite regions to be treated algebraically, through the sum of signed contributions. This additive structure will be used in the next section for the case of simple polygons, naturally leading to the Shoelace Formula as a global expression for calculating areas from vertices.

3 Shoelace Formula for Calculating the Area of a Polygon

As seen previously, the area of a triangle can be obtained from the coordinates of its vertices through determinants, as presented in Larson [7]. The extension of this procedure to polygons can be carried out by decomposing the region into triangles, so that the total area is obtained by summing the areas of the constituent parts. Although conceptually simple, this method becomes increasingly inefficient as the number of vertices of the polygon increases, requiring a large number of intermediate calculations.

In this context, the Shoelace Formula stands out as a direct alternative for calculating the area of simple polygons from the ordered coordinates of their vertices. Motivated by Santos [10], this formula makes it possible to reduce the problem to elementary algebraic operations involving only the polygon's vertices.

From a theoretical standpoint, this result can be interpreted as a consequence of Green's Theorem, as presented in Thomas [13], by transforming a line integral along the polygon's boundary into a discrete expression involving its coordinates. This interpretation highlights the deep relationship between Analytic Geometry, determinants, and Integral Calculus.

As highlighted by Bourke [3], the systematic organization of the polygon's vertices makes it possible to eliminate intermediate calculation steps, making the method particularly efficient for polygons with a large number of sides. Beyond its operational simplicity, the Shoelace Formula reveals an interesting connection between different areas of Mathematics, especially Linear Algebra and Vector Calculus.

Definition 3.1. Let P be a simple polygon in the Cartesian plane with ordered vertices $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, where $(x_{n+1}, y_{n+1}) = (x_1, y_1)$ to close the polygon.

The area of P can be calculated using the Shoelace Formula, given by

$$A(P) = \frac{1}{2}\left|\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i)\right|.$$

The Shoelace Formula is a consequence of Green's Theorem, stated as follows:

Theorem 3.2 (Green's Theorem (flux-divergence form))**.** *Let C be a simple closed curve, positively oriented, and let R be the region bounded by C. Let $F = Mi + Nj$ be a vector field whose functions M and N have continuous partial derivatives in an open region containing R. Then the outward flux of F along C equals the double integral of the divergence of F over R, that is,*

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \oint_C M\,dy - N\,dx.$$

The derivation of this method relates to line integrals and area integrals. A particular choice of vector field in Green's Theorem allows us to obtain a direct expression for calculating the area of plane regions. This choice leads to the proof of the following corollary:

Corollary 3.3. *If C is a simple closed curve, positively oriented, bounding a region R, then the area of R can be expressed as*

$$\text{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Proof. Consider the vector field $F = Mi + Nj$, defined by

$$M(x, y) = -\frac{1}{2}y \qquad \text{and} \qquad N(x, y) = \frac{1}{2}x.$$

Applying Green's Theorem in its classical form:

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

we have

$$\frac{\partial N}{\partial x} = \frac{1}{2} \qquad \text{and} \qquad \frac{\partial M}{\partial y} = -\frac{1}{2}.$$

Hence,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

Therefore,

$$\oint_C M\,dx + N\,dy = \iint_R 1\,dx\,dy.$$

Using the fact that $\text{Area}(R) = \iint_R dx\,dy$ and substituting M and N into the line integral, we obtain

$$\begin{aligned}\text{Area}(R) &= \oint_C \left(-\frac{1}{2}y\right) dx + \left(\frac{1}{2}x\right) dy \\ &= \frac{1}{2}\oint_C x\,dy - y\,dx.\end{aligned}$$

This yields the integral expression for calculating the area of a plane region bounded by a closed curve C, as presented in Thomas [13]. ☐

Moving on to the derivation of the Shoelace Formula from Green's Theorem, we will transform the line integral into a sum over the polygon's vertices. To do this, the polygon's boundary will be decomposed into line segments, each associated with a consecutive edge. By parametrizing each edge of the polygon, that is, each line segment $[(x_i, y_i), (x_{i+1}, y_{i+1})]$, we obtain a linear representation of the coordinates as a function of a parameter $t \in [0, 1]$. More precisely, each edge of the polygon can be described by $\gamma_i(t) = (x(t), y(t)) = (x_i + t(x_{i+1} - x_i),\ y_i + t(y_{i+1} - y_i))$. Now, differentiating $\gamma_i(t)$ with respect to t and applying the chain rule, we have

$$dx = (x_{i+1} - x_i)\,dt \qquad \text{and} \qquad dy = (y_{i+1} - y_i)\,dt.$$

Working now with the line integral, taking each $\gamma_i(t)$ as the curve C, we obtain

$$\int_{\gamma_i} (x\,dy - y\,dx) = \int_0^1 \Big[(x_i + t(x_{i+1} - x_i))(y_{i+1} - y_i) - (y_i + t(y_{i+1} - y_i))(x_{i+1} - x_i)\Big]\,dt.$$

Taking into account that this and noting the cancellation of the terms that depend on t, it follows that

$$\int_{\gamma_i} (x\,dy - y\,dx) = (x_i y_{i+1} - x_{i+1} y_i).$$

Summing over all paths $\gamma_i(t)$, that is, over the polygon's edges, we obtain

$$\oint_C (x\,dy - y\,dx) = \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i).$$

Therefore, by Green's Theorem,

$$A(P) = \frac{1}{2} \oint_C (x\,dy - y\,dx) = \frac{1}{2} \left| \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i) \right|.$$

The expression $(x_i y_{i+1} - x_{i+1} y_i)$ can be interpreted as the second-order determinant associated with the position vectors of consecutive points of the polygon, that is,

$$x_i y_{i+1} - x_{i+1} y_i = \begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}.$$

Thus, the area of the polygon can be written as the sum of the determinants associated with its edges, that is,

$$A = \frac{1}{2} \sum_{i=1}^{n} \begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}.$$

Expanding each determinant, we obtain

$$A = \frac{1}{2} \left[(x_1 y_2 + x_2 y_3 + \cdots + x_n y_1) - (y_1 x_2 + y_2 x_3 + \cdots + y_n x_1) \right].$$

On the other hand, this expression can be rearranged cyclically into a single determinant-like structure involving the polygon's vertices, resulting in the compact form

$$A = \frac{1}{2} \left| \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i) \right|.$$

Thus, we obtain the Shoelace Formula for calculating the area of a simple polygon.

The name "shoelace formula" arises precisely from the pattern formed by the cross products of the coordinates. By connecting the terms diagonally, as indicated by the arrows in the scheme below, the products interlace in a manner similar to a shoelace being tied.

In practical terms, the procedure can be understood in three simple steps:

1. Arrange the coordinates of the vertices in two rows;
2. Repeat the first point at the end;
3. Multiply diagonally, adding the products of the diagonals going from left to right (the blue arrows) and subtracting the sum of the products of the diagonals going from right to left (the arrows shown in red).

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n & x_1 \\ y_1 & y_2 & y_3 & \cdots & y_n & y_1 \end{vmatrix}$$

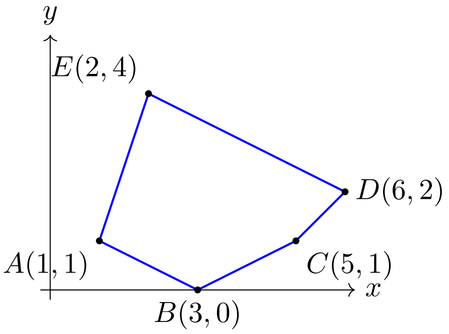
This structure shows that the Shoelace Formula results from a sum of second-order determinants formed by consecutive vertices of the polygon. This formula makes it possible to calculate areas directly from the coordinates of the points, avoiding geometric decompositions and making the procedure more systematic.

In the example that follows, we present the application of the Shoelace Formula to calculate the area of a simple polygon, illustrating the arrangement of the coordinates and the cross-product pattern between consecutive vertices.

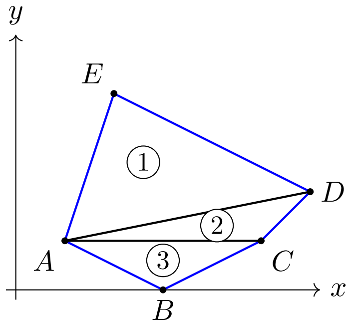
Example 3.4. Consider the polygon with vertices

$$A(1,1),\ B(3,0),\ C(5,1),\ D(6,2),\ E(2,4).$$

The polygon with the given coordinates is represented as follows



First, we will calculate the area of this polygon by dividing it into three triangles, computing their areas, and adding them together to obtain the total area of the polygon. Thus, we have



$$
\begin{aligned}
\text{Area} &= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 6 & 2 & 1 \\ 2 & 4 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 5 & 1 & 1 \\ 6 & 2 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \\ 5 & 1 & 1 \end{vmatrix} \\
\text{Area} &= \frac{1}{2}\cdot 14 + \frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 4 \\
\text{Area} &= 7 + 2 + 2 \\
\text{Area} &= 11\text{a.u}
\end{aligned}
$$

Now, applying the Shoelace Formula, we have

$$
\text{Area} = \frac{1}{2}\left| \begin{matrix} 1 & 3 & 5 & 6 & 2 & 1 \\ 1 & 0 & 1 & 2 & 4 & 1 \end{matrix} \quad \right|
$$

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}\left|(0 + 3 + 10 + 24 + 2) - (3 + 0 + 6 + 4 + 4)\right| \\
\text{Area} &= \frac{1}{2}(39 - 17) = \frac{1}{2}\cdot 22 = 11\text{a.u}
\end{aligned}
$$

Note that, while the decomposition into triangles requires intermediate steps, the Shoelace Formula provides the result directly from the coordinates of the vertices.

We now present an application of the Shoelace Formula to estimate the area of the State of Tocantins, based on an approximate polygonal representation of its boundary in the Cartesian plane.

As a final application of the results developed throughout this work, we consider the use of the Shoelace Formula to estimate the areas of plane regions with complex boundaries. In particular, a didactic application of geographic interest consists of approximating the area of federative units from polygonal representations of their boundaries.

From this perspective, the following subsection presents an application of the methodology to the State of Tocantins, illustrating how discretizing its boundary into points in the Cartesian plane allows an area estimate to be obtained through elementary algebraic operations.

3.0.1 Area of the State of Tocantins

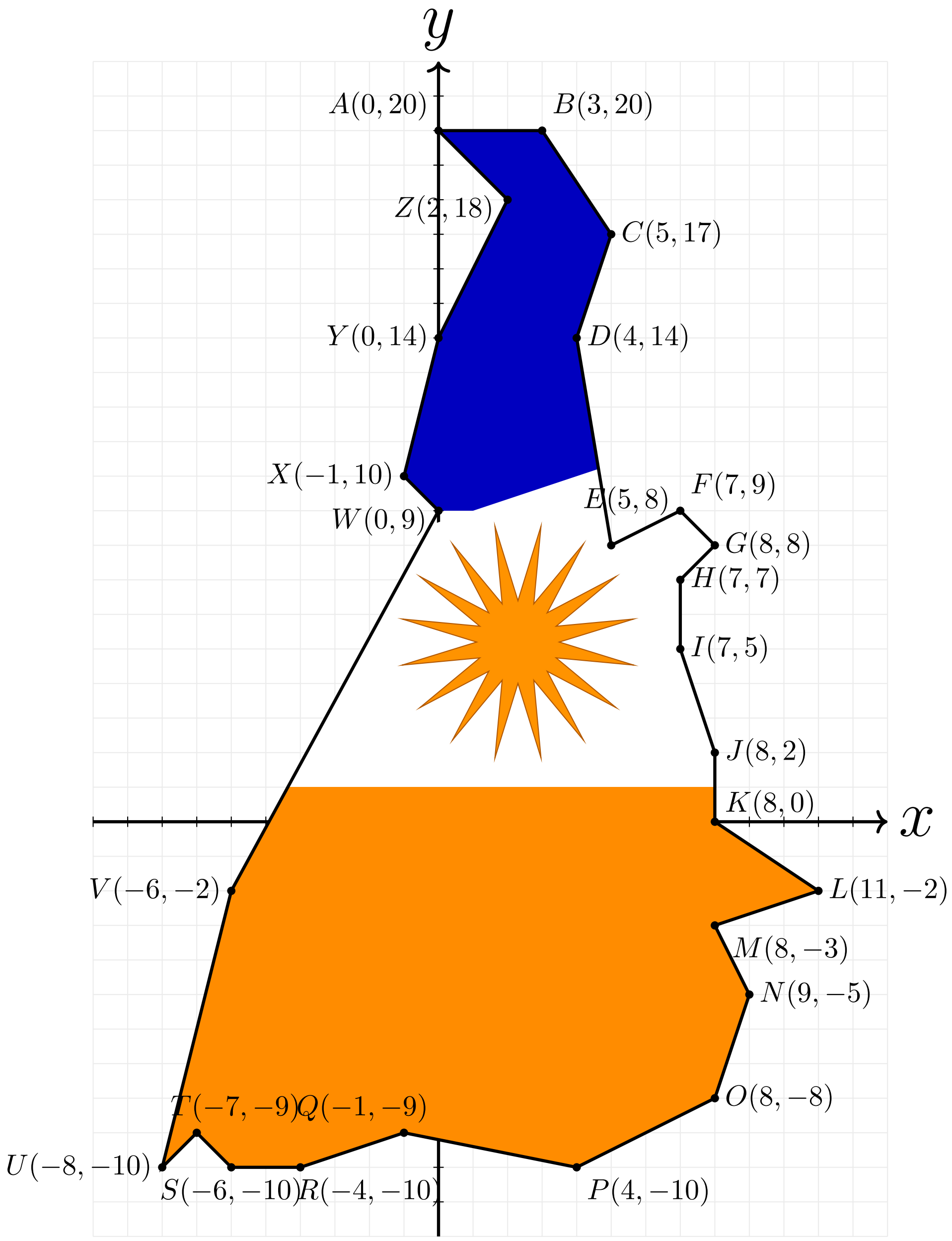
In this subsection, we present an application of the Shoelace Formula to estimate the area of the State of Tocantins, based on an approximate polygonal representation of its boundary in the Cartesian plane.

The central idea is to represent the state's boundary by means of an ordered set of points in the plane, so that the region bounded by these points can be treated as a simple polygon. This discretization makes it possible to replace a continuous geometric problem with a discrete algebraic problem, in which the area can be obtained directly through cross products of the coordinates of the vertices.

Below, we present the points used in the modeling and the direct application of the Shoelace Formula to calculate the approximate area.

Example 3.5. Consider the points $A(0,20), B(3,20), C(5,17), D(4,14), E(5,8), F(7,9), G(8,8), H(7,7)$ $I(7,5), J(8,2), K(8,0), L(11,-2), M(8,-3), N(9,-5), O(8,-8), P(4,-10), Q(-1,-9), R(-4,-10)$ $S(-6,-10), T(-7,-9), U(-8,-10), V(-6,-2), W(0,9), X(-1,10), Y(0,14), Z(2,18),$ which define a polygonal approximation of the boundary of the State of Tocantins.

We wish to estimate its area using the Shoelace Formula.



For the calculation, the vertices are arranged in cyclic order, and the Shoelace Formula is applied directly, summing the cross products of consecutive coordinates. Applying the Shoelace Formula, we obtain

$$\text{Area} = \frac{1}{2}\left|\begin{array}{ccccccccccccccccccccccccccc} 0 & 3 & 5 & 4 & 5 & 7 & 8 & 7 & 7 & 8 & 8 & 11 & 8 & 9 & 8 & 4 & -1 & -4 & -6 & -7 & -8 & -6 & 0 & -1 & 0 & 2 & 0 \\ 20 & 20 & 17 & 14 & 8 & 9 & 8 & 7 & 5 & 2 & 0 & -2 & -3 & -5 & -8 & -10 & -9 & -10 & -10 & -9 & -10 & -2 & 9 & 10 & 14 & 18 & 20 \end{array}\right|$$

$$\begin{aligned} Area &= \frac{1}{2}\left|244 - 799\right| = \frac{1}{2}\left|-555\right| \\ &= \frac{1}{2}\cdot 555 \\ &= 277.5\ a.u \end{aligned}$$

The value obtained provides a good approximation of the actual area of the State of Tocantins, whose official value is approximately $277,621$ km^2. This result highlights the efficiency of the Shoelace Formula as a tool for estimating areas of complex regions from a polygonal discretization.

4 Conclusion

The Shoelace Formula is usually presented as an algorithmic procedure for calculating the area of simple polygons, especially in contexts of Analytic Geometry and computational applications. However, its mathematical foundation is not always explored in teaching materials, which can limit conceptual understanding of the method.

In this work, we presented a derivation of the Shoelace Formula from Green's Theorem, showing that this expression is not merely an operational rule, but a direct consequence of results from Vector Calculus. To this end, we first explored the geometric interpretation of determinants in the area of triangles and then the parametrization of the edges of a simple polygon, which allowed the formula to be obtained systematically.

The application to the boundary of the State of Tocantins illustrated the efficiency of the method in situations involving a large number of vertices, providing an estimate consistent with the actual area of the region. This example reinforces the potential of the Shoelace Formula as a mathematical modeling tool and as a didactic resource for integrating Linear Algebra, Analytic Geometry, and Calculus.

It is hoped that this work will contribute to the education of teachers and students, broadening the theoretical understanding behind procedures frequently used in the classroom.

As future perspectives, we suggest a deeper analysis of Green's Theorem and its geometric applications, as well as the exploration of extensions of the Shoelace Formula to more general curves and area-calculation problems in more complex contexts.

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