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First-order expansions of coupled Leonardo sequences

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Abstract

In this paper, we investigate the sensitivity of coupled Leonardo sequences under perturbations of the recurrence coefficients and the coupling term. By introducing a perturbation parameter, we derive an explicit first-order asymptotic expansion and establish a linear sensitivity equation governing the leading correction term. Under suitable spectral assumptions, we obtain uniform error estimates and show that the perturbed sequence admits a first-order approximation whose accuracy is controlled by the perturbation magnitude. These results provide a quantitative description of the stability and local behavior of coupled Leonardo sequences and contribute to the perturbation theory of linear recursive systems.

Keywords: Coupled Leonardo sequence; perturbation; stability; spectral radius.

MSC 2020: 39A10; 39A30; 11B39.

Contents

1 Introduction

The Leonardo sequence, listed as $A001595$ in the On-Line Encyclopedia of Integer Sequences (OEIS) [1], is a classical example of a second-order linear recurrence closely related to the Fibonacci sequence. The Leonardo sequence is defined by

$$L_{n+2} = L_{n+1} + L_n + 1, \quad L_0 = L_1 = 1, \quad n \in \mathbb{N}.$$

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The relationship between the Fibonacci sequence $\{F_n\}_{n\geq 0}$ ($A000045$ in the OEIS [1]) and the Leonardo sequence $\{L_n\}_{n\geq 0}$ is given by

$$L_n = 2(F_n + F_{n-1}) - 1 = 2F_{n+1} - 1.$$

In [2], the coupled Leonardo sequence was introduced as a generalization of Leonardo-type sequences associated with a second-order Horadam recurrence. In its basic form, the sequence is defined by

$$L_n^{(k,t)} = kL_{n-1}^{(k,t)} + tL_{n-2}^{(k,t)} + Q(k,t), \qquad n \geq 2,$$

with initial conditions $L_0^{(k,t)} = L_1^{(k,t)} = 1$ and $k,\, t \in \mathbb{N}$, $kt \neq 0$. Equation (1) preserves the second-order structure of Horadam-type recurrences while introducing a coupling function $Q(k,t)$.

The purpose of the present paper is to propose a different type of generalization. Instead of increasing the order of the recurrence or introducing a non-constant term, we study the stability of the coupled Leonardo sequence under perturbations of its coefficients and coupling term. More precisely, we consider

$$L_n^{(\varepsilon)}(k,t) = (k+\varepsilon f(k,t))L_{n-1}^{(\varepsilon)} + (t+\varepsilon g(k,t))L_{n-2}^{(\varepsilon)} + \underbrace{Q(k,t)+\varepsilon h(k,t)}_{Q_\varepsilon(k,t)}, \qquad n \geq 2, \tag{1}$$

where $f, g, h : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ are non-vanishing functions, $\varepsilon \in \mathbb{R}$ is a small perturbation parameter, and

$$L_0^{(\varepsilon)} = a, \qquad L_1^{(\varepsilon)} = b,$$

with $a, b \in \mathbb{N}$.

The formulation of equation (1) has several advantages. In particular, it preserves the second-order structure of the original sequence and allows one to study how the sequence varies under small perturbations of the parameters.

The paper is organized as follows. In Section 2 we study the equilibrium solution and the spectral properties associated with the perturbed recurrence. In Section 3 we establish stability estimates with respect to the perturbation parameter. Section 4 is devoted to the sensitivity sequence and to the derivation of the first-order asymptotic expansion. Finally, Section 5 discusses the instability regime when the spectral radius is greater than one.

2 Equilibrium and spectral analysis

To analyze the perturbed recurrence (1), we begin by identifying its equilibrium solution and reducing the non-homogeneous equation to a homogeneous one. This transformation allows the dynamics to be expressed in terms of a linear recurrence with perturbed coefficients, whose behavior is determined by its characteristic roots.

Lemma 2.1. *Assume that,* $1-k-t-\varepsilon(f(k,t)+g(k,t)) \neq 0$. *Then the perturbed recurrence* (1) *admits the constant solution*

$$L_\varepsilon^* = \frac{Q_\varepsilon(k,t)}{1-k-t-\varepsilon\big(f(k,t)+g(k,t)\big)}. \tag{2}$$

Proof. Substituting the constant sequence $L_n^{(\varepsilon)} \equiv L_\varepsilon^*$ into (1), we obtain

$$L_\varepsilon^* = \big(k+\varepsilon f(k,t)\big)L_\varepsilon^* + \big(t+\varepsilon g(k,t)\big)L_\varepsilon^* + Q_\varepsilon(k,t).$$

Hence,

$$[1-k-t-\varepsilon(f(k,t)+g(k,t))]L_\varepsilon^* = Q_\varepsilon(k,t),$$

which proves the expression for L_ε^*. ☐

Note that if we consider

$$U_n^{(\varepsilon)} = L_n^{(\varepsilon)}(k,t) - L_\varepsilon^*, \tag{3}$$

we obtain

$$U_n^{(\varepsilon)} = (k+\varepsilon f(k,t))U_{n-1}^{(\varepsilon)} + (t+\varepsilon g(k,t))U_{n-2}^{(\varepsilon)}. \tag{4}$$

The introduction of the sequence $U_n^{(\varepsilon)}$ removes the non-homogeneous term from the recurrence and reduces the problem to a homogeneous linear relation with perturbed coefficients. This decomposition separates the equilibrium component from the dynamical behavior.

Although the sequence $U_n^{(\varepsilon)}$ will be used to control the dynamical component of the recurrence, our main object remains the perturbed coupled Leonardo sequence (1). Since

$$L_n^{(\varepsilon)} = U_n^{(\varepsilon)} + L_\varepsilon^*, \tag{5}$$

the stability of $L_n^{(\varepsilon)}$ follows from the joint control of the homogeneous component $U_n^{(\varepsilon)}$ and of the equilibrium L_ε^*.

The homogeneous recurrence associated with $U_n^{(\varepsilon)}$ has characteristic polynomial

$$p_\varepsilon(x) = x^2 - (k+\varepsilon f(k,t))x - (t+\varepsilon g(k,t)). \tag{6}$$

Let $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$ be its roots. We define the spectral radius by

$$r(\varepsilon) = \max\{|\rho_1(\varepsilon)|, |\rho_2(\varepsilon)|\}. \tag{7}$$

Proposition 2.2. *The condition $r(\varepsilon) < 1$ holds if, and only if*

$$|t+\varepsilon g(k,t)| < 1,$$

$$1-k-t-\varepsilon\big(f(k,t)+g(k,t)\big) > 0,$$

and

$$1+k-t+\varepsilon\big(f(k,t)-g(k,t)\big) > 0.$$

Proof. The result follows from the Jury stability criterion for second-order polynomials. More precisely, the roots of a polynomial of the form

$$x^2 - Ax - B$$

lie inside the unit disk (stability region) if and only if

$$|B| < 1, \qquad 1-A-B > 0, \qquad 1+A-B > 0;$$

see, for instance, [4]. Applying this criterion with

$$A = k+\varepsilon f(k,t), \qquad B = t+\varepsilon g(k,t),$$

yields the desired conditions. ☐

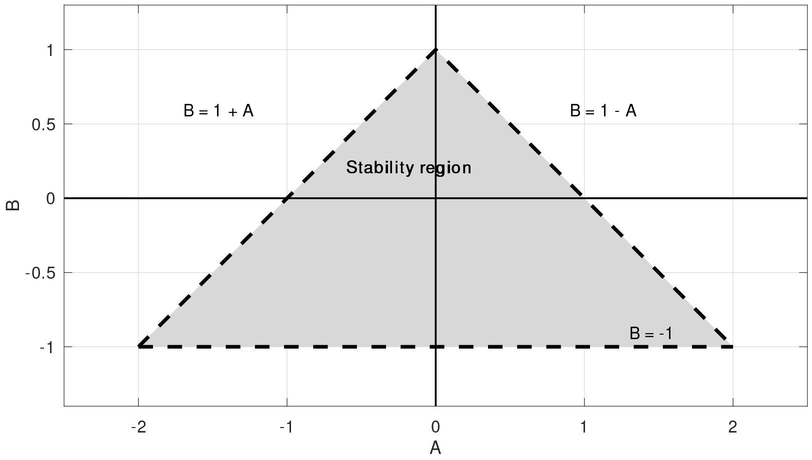

Figure 1: Stability region for $x^2 - Ax - B = 0$.

The stability region can be visualized in the (A, B)-plane, where $A = k + \varepsilon f(k,t)$ and $B = t + \varepsilon g(k,t)$. In this setting, the condition $r(\varepsilon) < 1$ corresponds to a triangular region, as illustrated in Figure 1.

Proposition 2.3 shows that, under the condition $r(\varepsilon) < 1$, the perturbed sequence (1) converges to its equilibrium solution (2).

Proposition 2.3. *Assume that the spectral radius $r(\varepsilon)$ associated with the homogeneous recurrence (5) satisfies $r(\varepsilon) < 1$. Then,*

$$L_n^{(\varepsilon)} \longrightarrow L_\varepsilon^* \qquad \text{as } n \to \infty.$$

Proof. From the Equation (3), we have

$$L_n^{(\varepsilon)} - L_\varepsilon^* = U_n^{(\varepsilon)}.$$

Since $r(\varepsilon) < 1$, we have

$$U_n^{(\varepsilon)} \to 0 \qquad \text{as } n \to \infty.$$

It follows that

$$L_n^{(\varepsilon)} \to L_\varepsilon^*,$$

which completes the proof.

□

3 Stability with respect to perturbations

In this section, we study the dependence of the perturbed recurrence on the parameter ε. In particular, we establish estimates describing the stability of the sequence under small perturbations of the coefficients and of the coupling term.

Proposition 3.1. *For each fixed $n \geq 0$, the mapping*

$$\varepsilon \mapsto L_n^{(\varepsilon)}$$

is a polynomial function of ε of degree at most $n - 1$.

Proof. We proceed by induction on n. For $n = 0, 1$, by the definition of the initial conditions in (1), the terms

$$L_0^{(\varepsilon)} = a, \qquad L_1^{(\varepsilon)} = b, \quad a, b \in \mathbb{N},$$

are constant functions of ε, hence polynomials.

Assume that $L_{n-1}^{(\varepsilon)}$ and $L_{n-2}^{(\varepsilon)}$ are polynomials in ε. Then, from the recurrence,

$$L_n^{(\varepsilon)} = (k + \varepsilon f(k,t))L_{n-1}^{(\varepsilon)} + (t + \varepsilon g(k,t))L_{n-2}^{(\varepsilon)} + Q(k,t) + \varepsilon h(k,t),$$

it follows that $L_n^{(\varepsilon)}$ is a linear combination of polynomials, and hence also a polynomial. The result follows by induction. ☐

Since the initial data do not depend on ε, the first-order variation starts only through the recurrence itself. This leads to an auxiliary nonhomogeneous recurrence driven by the unperturbed sequence. Corollary 3.2 follows immediately from the polynomial dependence.

Corollary 3.2. *For each fixed* $n \geq 0$, *one has*

$$\lim_{\varepsilon \to 0} L_n^{(\varepsilon)} = L_n^{(0)}.$$

Proof. The result follows immediately from Proposition 3.1, since for each fixed n, the mapping $\varepsilon \mapsto L_n^{(\varepsilon)}$ is continuous. ☐

Theorem 3.3 provides a uniform control of the perturbation with respect to both the parameter ε and the index n.

Theorem 3.3. *Assume that* $r(0) < 1$. *Then there exist constants* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, for every* $|\varepsilon| < \varepsilon_0$,

$$\sup_{n \geq 0} |L_n^{(\varepsilon)} - L_n^{(0)}| \leq C|\varepsilon|.$$

Proof. By equation (5), we have

$$L_n^{(\varepsilon)} = U_n^{(\varepsilon)} + L_\varepsilon^*, \qquad L_n^{(0)} = U_n^{(0)} + L_0^*.$$

Hence,

$$|L_n^{(\varepsilon)} - L_n^{(0)}| \leq |U_n^{(\varepsilon)} - U_n^{(0)}| + |L_\varepsilon^* - L_0^*|.$$

Since $r(0) < 1$, by continuity of the roots there exist $\varepsilon_0 > 0$ and $0 < \theta < 1$ such that $r(\varepsilon) \leq \theta$ for all $|\varepsilon| < \varepsilon_0$. Thus the homogeneous solutions satisfy a uniform exponential bound: there exists $C_1 > 0$ such that

$$|U_n^{(\varepsilon)}| \leq C_1\theta^n, \qquad |U_n^{(0)}| \leq C_1\theta^n, \quad \forall n \geq 0, \ |\varepsilon| < \varepsilon_0.$$

Set

$$V_n = U_n^{(\varepsilon)} - U_n^{(0)}.$$

Using equation (4), we obtain

$$V_n = (k + \varepsilon f(k,t))V_{n-1} + (t + \varepsilon g(k,t))V_{n-2} + \varepsilon f(k,t)U_{n-1}^{(0)} + \varepsilon g(k,t)U_{n-2}^{(0)}.$$

Since $r(\varepsilon) \leq \theta < 1$, the homogeneous recurrence associated with V_n is uniformly exponentially stable. Moreover, the forcing term is of order $O(\varepsilon\theta^n)$. Standard estimates for stable linear recurrences then imply the existence of a constant $C_2 > 0$ such that

$$\sup_{n \geq 0} |U_n^{(\varepsilon)} - U_n^{(0)}| \leq C_2|\varepsilon|.$$

Moreover, from the explicit formula for L_ε^* and the condition $1-k-t \neq 0$, the map

$$\varepsilon \mapsto L_\varepsilon^*$$

is differentiable in a neighborhood of 0. Hence, there exists $C_3 > 0$ such that

$$|L_\varepsilon^* - L_0^*| \leq C_3|\varepsilon|.$$

Combining the previous estimates, we obtain

$$\sup_{n\geq 0} |L_n^{(\varepsilon)} - L_n^{(0)}| \leq (C_2 + C_3)|\varepsilon|.$$

This completes the proof.

□

4 Sensitivity analysis and first-order expansion

The study of sensitivity with respect to parameters plays an important role in the analysis of discrete dynamical systems. In the context of linear recurrences, it provides a quantitative description of how solutions vary under small perturbations of the coefficients and coupling terms, as considered in equation (1). This type of analysis is closely related to perturbation theory and is used to assess the stability and robustness of discrete models.

The main purpose of this section is to derive the first-order expansion of the coupled Leonardo sequence under parameter perturbations and to identify the auxiliary recurrence governing its sensitivity (see [3, 4]).

Definition 4.1. The sensitivity sequence $\{Z_n\}_{n\geq 0}$ is defined by

$$Z_n = \frac{d}{d\varepsilon} L_n^{(\varepsilon)}\bigg|_{\varepsilon=0}, \qquad n \geq 0.$$

The following result provides a recurrence satisfied by the sensitivity sequence Z_n.

Proposition 4.2. *The sensitivity sequence* $\{Z_n\}_{n\geq 0}$ *satisfies*

$$Z_0 = 0, \qquad Z_1 = 0,$$

and, for $n \geq 2$,

$$Z_n = kZ_{n-1} + tZ_{n-2} + f(k,t)\, L_{n-1}^{(0)} + g(k,t)\, L_{n-2}^{(0)} + h(k,t). \tag{8}$$

Proof. Differentiating the perturbed recurrence given in equation (1) with respect to ε, we obtain

$$\frac{d}{d\varepsilon} L_n^{(\varepsilon)} = f(k,t)L_{n-1}^{(\varepsilon)} + (k+\varepsilon f(k,t))\frac{d}{d\varepsilon}L_{n-1}^{(\varepsilon)} + g(k,t)L_{n-2}^{(\varepsilon)} + (t+\varepsilon g(k,t))\frac{d}{d\varepsilon}L_{n-2}^{(\varepsilon)} + h(k,t).$$

Evaluating at $\varepsilon = 0$, and using the definition 4.1, yields

$$Z_n = kZ_{n-1} + tZ_{n-2} + f(k,t)L_{n-1}^{(0)} + g(k,t)L_{n-2}^{(0)} + h(k,t).$$

Since the initial conditions $L_0^{(\varepsilon)} = a$ and $L_1^{(\varepsilon)} = b$ do not depend on ε, it follows that

$$Z_0 = \frac{d}{d\varepsilon} L_0^{(\varepsilon)}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} a = 0, \qquad Z_1 = \frac{d}{d\varepsilon} L_1^{(\varepsilon)}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} b = 0.$$

□

The recurrence satisfied by Z_n shows that the sensitivity sequence is driven by the unperturbed dynamics $L_n^{(0)}$. In particular, the first-order response inherits the spectral structure of the original recurrence while incorporating the perturbed terms associated with f, g, and h.

Theorem 4.3 shows that the sequence Z_n provides the first-order term in the asymptotic expansion of $L_n^{(\varepsilon)}$.

Theorem 4.3. *For each fixed $n \geq 0$, one has*

$$L_n^{(\varepsilon)} = L_n^{(0)} + \varepsilon Z_n + O(\varepsilon^2), \qquad \varepsilon \to 0. \tag{9}$$

Proof. Fix $n \geq 0$. By Proposition 3.1, the mapping

$$\varepsilon \mapsto L_n^{(\varepsilon)}$$

is a polynomial function of ε. In particular, it is twice continuously differentiable in a neighborhood of $\varepsilon = 0$. By Taylor's formula with remainder, there exists ξ_ε between 0 and ε such that

$$L_n^{(\varepsilon)} = L_n^{(0)} + \varepsilon \left.\frac{d}{d\varepsilon} L_n^{(\varepsilon)}\right|_{\varepsilon=0} + \frac{\varepsilon^2}{2} \left.\frac{d^2}{d\varepsilon^2} L_n^{(\varepsilon)}\right|_{\varepsilon=\xi_\varepsilon}.$$

By the definition 4.1,

$$Z_n = \left.\frac{d}{d\varepsilon} L_n^{(\varepsilon)}\right|_{\varepsilon=0}.$$

Hence,

$$L_n^{(\varepsilon)} = L_n^{(0)} + \varepsilon Z_n + \frac{\varepsilon^2}{2} \left.\frac{d^2}{d\varepsilon^2} L_n^{(\varepsilon)}\right|_{\varepsilon=\xi_\varepsilon}.$$

Since $L_n^{(\varepsilon)}$ is a polynomial in ε, its second derivative is continuous and therefore bounded in some interval $|\varepsilon| \leq \varepsilon_0$. Thus, there exists a constant $C_n > 0$ such that

$$\left| \left.\frac{d^2}{d\varepsilon^2} L_n^{(\varepsilon)}\right|_{\varepsilon=\xi_\varepsilon} \right| \leq C_n,$$

for all sufficiently small ε. Consequently,

$$\left| L_n^{(\varepsilon)} - L_n^{(0)} - \varepsilon Z_n \right| \leq \frac{C_n}{2} \varepsilon^2,$$

which proves that

$$L_n^{(\varepsilon)} = L_n^{(0)} + \varepsilon Z_n + O(\varepsilon^2), \qquad \varepsilon \to 0.$$

□

The representation of the recurrence (9) separates the dependence on ε from the intrinsic dynamics of the sequence. In particular, $L_n^{(0)}$ describes the unperturbed behavior, while Z_n captures the first-order response to perturbations.

Remark 4.4. The expansion obtained in Theorem 4.3 is pointwise with respect to the index n. In general, the constant involved in the $O(\varepsilon^2)$ term may depend on n. Uniform first-order expansions require additional stability assumptions, such as a spectral radius strictly smaller than one.

5 Instability

In Sections 3 and 4, it was shown that when the spectral radius is strictly less than one, the effect of perturbations remains controlled. In the present section, we analyze the complementary case in which the spectral radius is greater than one.

Proposition 5.1. *Assume that* $r(0) > 1$. *Suppose that the unperturbed characteristic polynomial* (6) *has a simple dominant root* ρ_0, *that is,*

$$|\rho_0| > \max\{1, |\sigma_0|\},$$

where σ_0 *denotes the other characteristic root. Assume also that the coefficient of the dominant mode in the homogeneous component of* $L_n^{(0)}$ *is nonzero and that*

$$\rho'(0) \neq 0,$$

where $\rho(\varepsilon)$ *denotes the continuation of the dominant root for* ε *near* 0. *Then the convergence*

$$L_n^{(\varepsilon)} \to L_n^{(0)}$$

as $\varepsilon \to 0$ *is not uniform in* n.

Proof. Let $U_n^{(\varepsilon)} = L_n^{(\varepsilon)} - L_\varepsilon^*$ (Equation (3)). By Equation (4), $U_n^{(\varepsilon)}$ satisfies the homogeneous recurrence

$$U_n^{(\varepsilon)} = (k + \varepsilon f(k,t))U_{n-1}^{(\varepsilon)} + (t + \varepsilon g(k,t))U_{n-2}^{(\varepsilon)}.$$

Let $\rho(\varepsilon)$ and $\sigma(\varepsilon)$ be the roots of the perturbed characteristic polynomial, with

$$\rho(0) = \rho_0, \qquad \sigma(0) = \sigma_0.$$

Since ρ_0 is a simple dominant root, for $|\varepsilon|$ sufficiently small the roots depend smoothly on ε, and $\rho(\varepsilon)$ remains dominant. Thus the homogeneous component can be written as

$$U_n^{(\varepsilon)} = C(\varepsilon)\rho(\varepsilon)^n + D(\varepsilon)\sigma(\varepsilon)^n,$$

where $C(\varepsilon)$ and $D(\varepsilon)$ depend on the initial conditions. By hypothesis,

$$C(0) \neq 0.$$

Since $\rho'(0) \neq 0$, for sufficiently small nonzero ε one has

$$\rho(\varepsilon) \neq \rho(0).$$

Moreover, since $|\rho_0| > 1$, we still have

$$|\rho(\varepsilon)| > 1$$

for ε small.

Now,

$$L_n^{(\varepsilon)} - L_n^{(0)} = (U_n^{(\varepsilon)} - U_n^{(0)}) + (L_\varepsilon^* - L_0^*).$$

The second term is independent of n, and therefore cannot compensate the growth of the homogeneous part.

Using the representations of $U_n^{(\varepsilon)}$ and $U_n^{(0)}$, we obtain

$$U_n^{(\varepsilon)} - U_n^{(0)} = C(\varepsilon)\rho(\varepsilon)^n - C(0)\rho_0^n + D(\varepsilon)\sigma(\varepsilon)^n - D(0)\sigma_0^n.$$

Since $\rho(\varepsilon)$ and ρ_0 are distinct dominant roots and $|\rho_0| > 1$, the dominant part

$$C(\varepsilon)\rho(\varepsilon)^n - C(0)\rho_0^n$$

does not remain bounded as $n \to \infty$, except in degenerate cases excluded by the assumptions. Hence

$$\sup_{n\geq 0} |L_n^{(\varepsilon)} - L_n^{(0)}| = +\infty$$

for sufficiently small nonzero ε. In particular,

$$\sup_{n\geq 0} |L_n^{(\varepsilon)} - L_n^{(0)}| \not\longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

Therefore, the convergence is not uniform in n. ☐

6 Concluding remarks

In this paper, we analyzed the stability of perturbed coupled Leonardo sequences under simultaneous perturbations of the recurrence coefficients and of the coupling term. The equilibrium structure of the recurrence was characterized through a spectral approach, leading to stability and instability criteria in terms of the spectral radius.

We also introduced the sensitivity sequence associated with the perturbation parameter and derived the first-order asymptotic expansion

$$L_n^{(\varepsilon)} = L_n^{(0)} + \varepsilon Z_n + O(\varepsilon^2),$$

which quantitatively describes the response of the sequence to small perturbations.

These results provide a framework for the analysis of coupled Leonardo-type recurrences under small perturbations and may be adapted to other families of linear recurrences with nonhomogeneous structure.

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